PHYSICAL REVIEW E **70**, 036302 (2004)

Reynolds stress model involving the mean spin tensor

Yu-Ning Huang[1,*] and Hui-Yang Ma[2,†]
[1]*State Key Laboratory for Turbulence and Complex Systems, Department of Mechanics and Engineering Science, Peking University, Beijing 100871, People's Republic of China*
[2]*Department of Physics, Graduate School of the Chinese Academy of Sciences, P. O. Box 3908, Beijing 100039, People's Republic of China*
(Received 16 November 2003; published 3 September 2004)

In this work, we develop a Reynolds stress model along the line of the approach presented by Huang [Commun. Nonlinear Sci. Numer. Simul. **9**, 543 (2004)], aiming to assess the role and contribution of the mean spin tensor in turbulence modeling. Here, the constitutive functional for the Reynolds stress depends on the mean spin tensor as well as the mean stretching tensor and its Jaumann derivative, the turbulent kinetic energy K, and the turbulent dissipation rate ε, which is at the complexity level of $p=1, m=1$, and $n=0$ of a rate-type constitutive equation for the Reynolds stress proposed in the aforementioned paper. The explicit form for the Reynolds stress is obtained with recourse to the representation theorem and the theory of invariants developed in modern rational continuum mechanics, and, as an approximation, a nonlinear cubic K-ε model is worked out in which the model coefficients are analytically identified based on the experimental results of Tavoularis and Corrsin [J. Fluid Mech. **104**, 311 (1981)]. In addition, numerical results based on this model, in the forms of employing the Jaumann derivative and the Oldroyd derivative, respectively, for homogeneous turbulent shear flow and fully developed turbulent flow over a backward-facing step, are presented in comparison with those obtained based on a few previously proposed linear and nonlinear K-ε models, showing reasonably good agreement with the experimental results and the DNS data concerned and a better performance than the previously developed quadratic models.

DOI: 10.1103/PhysRevE.70.036302 PACS number(s): 47.27.Eq, 46.05.+b, 47.27.Jv, 47.27.Nz

I. INTRODUCTION

In a note Rivlin [1] pointed out that there exists an analogy between the constitutive equation for the turbulent flows of a Newtonian fluid and that for the laminar flows of non-Newtonian fluids. Indeed, as seen over the past four decades, a great deal of research concerned indicates that in many aspects the turbulent flow of a Newtonian fluid behaves like non-Newtonian fluids, e.g., showing nonlinear viscoelasticity and fading memory of its own history (see, e.g., Huang [2], Tavoularis and Corrsin [3], Liepmann [4], and Proudman [5]). However, there were controversies concerning the validity of the principle of material frame-indifference in continuum mechanics when applied to turbulence modeling. By examining a steady homogeneous turbulent plane strain in a steadily rotating framework based on a second-order closure model, Lumley [6] showed that the effect of rotation on the Reynolds stress is serious and concluded that the principle of material frame-indifference is dramatically violated in turbulence and thus should be discarded in turbulence modeling. Recently, in a work aiming to clarify and set straight the controversies concerned in modeling the Reynolds stress, it has been shown by Huang and Durst [7] that, due to being subjected to the *dynamical process* induced adscititiously from taking the ensemble average on the Navier-Stokes equations, in fact, the invariance group of the Reynolds stress is the extended Galilean group of transformations rather than the Euclidean group of transformations as taken before from the perspective of kinematics analysis; and, as a result, unlike modeling the non-Newtonian fluids in which case the constitutive equations exclude the spin tensor from being a constitutive variable in accord with the principle of material frame-indifference in continuum mechanics, the frame-dependent kinematical quantities, e.g., the mean spin tensor, may be allowed to play an effective role in turbulence modeling. Therefore, in the sense of being in conformity with the averaged Navier-Stokes equations, a number of turbulence closure models proposed hitherto, for examples, the model of Pope [8], the model of Gatski and Speziale [9], the model of Shih *et al.* [10], and the model of Craft *et al.* [11], are all physically justified to include the *mean spin tensor* as a constitutive argument. Of course, there are relevant mathematical constraints to be observed, such as the realizability (see Schumann [12] and Lumley [13]).

In this work, we shall develop a Reynolds stress model involving the mean spin tensor along the line of the approach presented in Huang [2], to further assess the role and contribution of the mean spin tensor in turbulence modeling. To this end, here, we consider a constitutive equation for the Reynolds stress $\boldsymbol{\tau}$ that depends on $\mathring{\boldsymbol{\tau}}$—the Jaumann derivative of the Reynolds stress, the mean stretching tensor $\mathbf{D}$ and its Jaumann derivative $\mathring{\mathbf{D}}$, and the mean spin tensor $\mathbf{W}$, along with the turbulent kinetic energy K and the turbulent dissipation rate ε. With recourse to the representation theorem and the theory of invariants developed in modern continuum mechanics (see Wang [14], Smith [15], and Spencer [16]), the explicit form of this constitutive equation is obtained and, furthermore, an approximate form is worked out in

*Electronic address: yuninghuang@yahoo.com
†Electronic address: hyma@163bj.com

which the model coefficients are identified based on the experimental results of Tavoularis and Corrsin [3]. In addition, to evaluate the role and contribution of the mean spin tensor in turbulence modeling, the numerical calculations based on this new nonlinear K-ε model in the forms of using the Jaumann derivative and the Oldroyd derivative, respectively, for homogeneous turbulent shear flow and fully developed turbulent flow over a backward-facing step will be presented in comparison with those obtained by employing a few previously proposed K-ε models.

II. A CONSTITUTIVE EQUATION FOR THE REYNOLDS STRESS INVOLVING THE MEAN SPIN TENSOR

A. Theoretical background

We consider the turbulent flows of an incompressible Newtonian fluid. The mean continuity equation and the Reynolds-averaged Navier-Stokes equations read, respectively,

$$\mathrm{div}\overline{\mathbf{v}} = 0, \tag{1}$$

$$\varrho \frac{D\overline{\mathbf{v}}}{Dt} = \mathrm{div}(\mathbf{T} - \varrho \boldsymbol{\tau}) + \varrho \mathbf{B}, \tag{2}$$

where an overbar represents the mean value of ensemble average, ϱ is the mass density (constant), $\mathbf{B}$ is the prescribed body force density, D/Dt denotes the material time derivative associated with the mean velocity field $\overline{\mathbf{v}}$, $\mathbf{T}=-\overline{p}\mathbf{1}+2\mu\mathbf{D}$, wherein $\mathbf{1}$ is the unit tensor, $\mathbf{D}=\frac{1}{2}[\mathrm{grad}\overline{\mathbf{v}}+(\mathrm{grad}\overline{\mathbf{v}})^T]$ is the mean stretching tensor, $\overline{p}$ is the mean pressure, μ is the viscosity, and $\boldsymbol{\tau}:=\overline{\mathbf{v}'\otimes\mathbf{v}'}$ is the Reynolds stress tensor, the modeling of which leads to the so-called *closure problem* in turbulence modeling.

Here, we follow the approach presented by Huang [2] to establish a constitutive equation for the Reynolds stress $\boldsymbol{\tau}$ involving the *mean spin tensor* $\mathbf{W}$—its constitutive equation takes the form

$$\boldsymbol{\tau} = \mathcal{F}(\mathring{\boldsymbol{\tau}}, \mathbf{D}, \mathring{\mathbf{D}}, \mathbf{W}; K, \varepsilon), \tag{3}$$

where $\mathcal{F}$ is the constitutive functional, $\circ$ denotes the Jaumann derivative, $\mathring{\boldsymbol{\tau}}=D\boldsymbol{\tau}/Dt+\boldsymbol{\tau}\mathbf{W}-\mathbf{W}\boldsymbol{\tau}$, $\mathring{\mathbf{D}}=D\mathbf{D}/Dt+\mathbf{D}\mathbf{W}-\mathbf{W}\mathbf{D}$ and $\mathbf{W}=\frac{1}{2}[\mathrm{grad}\overline{\mathbf{v}}-(\mathrm{grad}\overline{\mathbf{v}})^T]$.

Equation (3) is a special case of a rate-type constitutive equation for the Reynolds stress proposed in Ref. [2], which reads

$$\mathcal{R}(\boldsymbol{\tau}, \mathring{\boldsymbol{\tau}}_1, \ldots, \mathring{\boldsymbol{\tau}}_p; \mathbf{D}, \mathring{\mathbf{D}}_1, \ldots, \mathring{\mathbf{D}}_m; \mathbf{W}, \mathring{\mathbf{W}}_1, \ldots, \mathring{\mathbf{W}}_n; K, \varepsilon) = \mathbf{0}, \tag{4}$$

i.e., at the *complexity level* of $p=1, m=1$, and $n=0$, where $\mathcal{R}$ is the constitutive functional.

Note that the constitutive equation (3) takes into account some history effects of both the Reynolds stress $\boldsymbol{\tau}$ and the mean stretching tensor $\mathbf{D}$, e.g., the relaxation effect of the Reynolds stress, by containing their Jaumann derivatives as constitutive variables. Clearly, with the mean spin tensor $\mathbf{W}$'s being taken to be a constitutive variable, the constitutive equation (3) includes as a special case the constitutive equation investigated in Ref. [2], which is

$$\boldsymbol{\tau} = \mathcal{L}(\mathring{\boldsymbol{\tau}}, \mathbf{D}, \mathring{\mathbf{D}}; K, \varepsilon). \tag{5}$$

It should be noted that, in the paper [2] on modeling the Reynolds stress in the context of continuum mechanics, it has been stressed that, although it was pointed out by Rivlin [1] that there exists a similarity between the constitutive equation for the turbulent flows of a Newtonian fluid and that for the laminar flows of non-Newtonian fluids, in reality, generally speaking, modeling the Reynolds stress appears to be much more involved and more complicated than modeling the Cauchy stress in the sense that the frame-dependent kinematical quantities, e.g., the mean spin tensor, may be allowed to play an effective role (see Refs. [2,6–8]) in turbulence modeling, whereas in modeling the Cauchy stress of a non-Newtonian fluid the frame-dependent kinematical quantities, such as the spin tensor, are excluded from being the constitutive variables to be in accordance with the principle of material frame-indifference in continuum mechanics.

B. The explicit form of the model and its approximations

By making use of the representation theorem and the theory of invariants of Wang [14], Smith [15], and Spencer [16], we obtain the following irreducible explicit form for the constitutive equation (3):

$$\begin{aligned}
\boldsymbol{\tau} &= \mathcal{F}(\mathring{\boldsymbol{\tau}}, \mathbf{D}, \mathring{\mathbf{D}}, \mathbf{W}; K, \varepsilon) \\
&= a_0\mathbf{1} + a_1\mathring{\boldsymbol{\tau}} + a_2\mathbf{D} + a_3\mathring{\mathbf{D}} + b_1(\mathring{\boldsymbol{\tau}})^2 + b_2\mathbf{D}^2 + b_3(\mathring{\mathbf{D}})^2 + b_4\mathbf{W}^2 + c_1[\mathring{\boldsymbol{\tau}}\mathbf{D} + \mathbf{D}\mathring{\boldsymbol{\tau}}] + c_2[(\mathring{\boldsymbol{\tau}})^2\mathbf{D} + \mathbf{D}(\mathring{\boldsymbol{\tau}})^2] \\
&\quad + c_3[\mathring{\boldsymbol{\tau}}\mathbf{D}^2 + \mathbf{D}^2\mathring{\boldsymbol{\tau}}] + c_4[(\mathring{\boldsymbol{\tau}})^2\mathbf{D}^2 + \mathbf{D}^2(\mathring{\boldsymbol{\tau}})^2] + d_1[\mathring{\boldsymbol{\tau}}\mathring{\mathbf{D}} + \mathring{\mathbf{D}}\mathring{\boldsymbol{\tau}}] + d_2[(\mathring{\boldsymbol{\tau}})^2\mathring{\mathbf{D}} + \mathring{\mathbf{D}}(\mathring{\boldsymbol{\tau}})^2] + d_3[\mathring{\boldsymbol{\tau}}(\mathring{\mathbf{D}})^2 + (\mathring{\mathbf{D}})^2\mathring{\boldsymbol{\tau}}] + d_4[(\mathring{\boldsymbol{\tau}})^2(\mathring{\mathbf{D}})^2 + (\mathring{\mathbf{D}})^2(\mathring{\boldsymbol{\tau}})^2] \\
&\quad + e_1[\mathbf{D}\mathring{\mathbf{D}} + \mathring{\mathbf{D}}\mathbf{D}] + e_2[\mathbf{D}^2\mathring{\mathbf{D}} + \mathring{\mathbf{D}}\mathbf{D}^2] + e_3[\mathbf{D}(\mathring{\mathbf{D}})^2 + (\mathring{\mathbf{D}})^2\mathbf{D}] + e_4[\mathbf{D}^2(\mathring{\mathbf{D}})^2 + (\mathring{\mathbf{D}})^2\mathbf{D}^2] + f_1[\mathring{\boldsymbol{\tau}}\mathbf{W} - \mathbf{W}\mathring{\boldsymbol{\tau}}] + f_2[\mathbf{W}\mathring{\boldsymbol{\tau}}\mathbf{W}] \\
&\quad + f_3[(\mathring{\boldsymbol{\tau}})^2\mathbf{W} - \mathbf{W}(\mathring{\boldsymbol{\tau}})^2] + f_4[\mathbf{W}(\mathring{\boldsymbol{\tau}})^2\mathbf{W}] + f_5[\mathbf{W}\mathring{\boldsymbol{\tau}}\mathbf{W}^2 - \mathbf{W}^2\mathring{\boldsymbol{\tau}}\mathbf{W}] + g_1[\mathbf{D}\mathbf{W} - \mathbf{W}\mathbf{D}] + g_2[\mathbf{W}\mathbf{D}\mathbf{W}] + g_3[\mathbf{D}^2\mathbf{W} - \mathbf{W}\mathbf{D}^2] \\
&\quad + g_4[\mathbf{W}\mathbf{D}^2\mathbf{W}] + g_5[\mathbf{W}\mathbf{D}\mathbf{W}^2 - \mathbf{W}^2\mathbf{D}\mathbf{W}] + h_1[\mathring{\mathbf{D}}\mathbf{W} - \mathbf{W}\mathring{\mathbf{D}}] + h_2[\mathbf{W}\mathring{\mathbf{D}}\mathbf{W}] + h_3[(\mathring{\mathbf{D}})^2\mathbf{W} - \mathbf{W}(\mathring{\mathbf{D}})^2] + h_4[\mathbf{W}(\mathring{\mathbf{D}})^2\mathbf{W}] \\
&\quad + h_5[\mathbf{W}\mathring{\mathbf{D}}\mathbf{W}^2 - \mathbf{W}^2\mathring{\mathbf{D}}\mathbf{W}],
\end{aligned} \tag{6}$$

where the coefficients $a_0, a_1, a_2, \ldots, h_4$, and h_5 are functions of K and ε as well as the following invariants (note that $\mathrm{tr}\mathbf{D} = \mathrm{tr}\mathring{\mathbf{D}} = 0$):

$$\mathrm{tr}\mathring{\boldsymbol{\tau}} = 2DK/Dt, \mathrm{tr}(\mathring{\boldsymbol{\tau}})^2, \mathrm{tr}\mathbf{D}^2, \mathrm{tr}(\mathring{\mathbf{D}})^2, \mathrm{tr}(\mathring{\boldsymbol{\tau}})^3, \mathrm{tr}\mathbf{D}^3, \mathrm{tr}(\mathring{\mathbf{D}})^3, \mathrm{tr}\mathbf{W}^2, \mathrm{tr}[\mathring{\boldsymbol{\tau}}\mathbf{D}], \mathrm{tr}[(\mathring{\boldsymbol{\tau}})^2\mathbf{D}], \mathrm{tr}[\mathring{\boldsymbol{\tau}}\mathbf{D}^2], \mathrm{tr}[(\mathring{\boldsymbol{\tau}})^2\mathbf{D}^2], \mathrm{tr}[\mathring{\boldsymbol{\tau}}\mathring{\mathbf{D}}], \mathrm{tr}[(\mathring{\boldsymbol{\tau}})^2\mathring{\mathbf{D}}], \mathrm{tr}[\mathring{\boldsymbol{\tau}}(\mathring{\mathbf{D}})^2],$$
$$\mathrm{tr}[(\mathring{\boldsymbol{\tau}})^2(\mathring{\mathbf{D}})^2], \mathrm{tr}[\mathbf{D}\mathring{\mathbf{D}}], \mathrm{tr}[\mathbf{D}^2\mathring{\mathbf{D}}], \mathrm{tr}[\mathbf{D}(\mathring{\mathbf{D}})^2], \mathrm{tr}[\mathbf{D}^2(\mathring{\mathbf{D}})^2], \mathrm{tr}[\mathring{\boldsymbol{\tau}}\mathbf{W}^2], \mathrm{tr}[(\mathring{\boldsymbol{\tau}})^2\mathbf{W}^2], \mathrm{tr}[(\mathring{\boldsymbol{\tau}})^2\mathbf{W}^2\mathring{\boldsymbol{\tau}}\mathbf{W}], \mathrm{tr}[\mathbf{D}\mathbf{W}^2], \mathrm{tr}[\mathbf{D}^2\mathbf{W}^2], \mathrm{tr}[\mathbf{D}^2\mathbf{W}^2\mathbf{D}\mathbf{W}],$$
$$\mathrm{tr}[\mathring{\mathbf{D}}\mathbf{W}^2], \mathrm{tr}[(\mathring{\mathbf{D}})^2\mathbf{W}^2], \mathrm{tr}[(\mathring{\mathbf{D}})^2\mathbf{W}^2\mathring{\mathbf{D}}\mathbf{W}], \mathrm{tr}[\mathring{\boldsymbol{\tau}}\mathbf{D}\mathring{\mathbf{D}}], \mathrm{tr}[\mathring{\boldsymbol{\tau}}\mathbf{D}\mathbf{W}], \mathrm{tr}[(\mathring{\boldsymbol{\tau}})^2\mathbf{D}\mathbf{W}], \mathrm{tr}[\mathring{\boldsymbol{\tau}}\mathbf{D}^2\mathbf{W}], \mathrm{tr}[\mathring{\boldsymbol{\tau}}\mathbf{W}^2\mathbf{D}\mathbf{W}], \mathrm{tr}[\mathring{\boldsymbol{\tau}}\mathring{\mathbf{D}}\mathbf{W}], \mathrm{tr}[(\mathring{\boldsymbol{\tau}})^2\mathring{\mathbf{D}}\mathbf{W}],$$
$$\mathrm{tr}[\mathring{\boldsymbol{\tau}}(\mathring{\mathbf{D}})^2\mathbf{W}], \mathrm{tr}[\mathring{\boldsymbol{\tau}}\mathbf{W}^2\mathring{\mathbf{D}}\mathbf{W}], \mathrm{tr}[\mathbf{D}\mathring{\mathbf{D}}\mathbf{W}], \mathrm{tr}[\mathbf{D}^2\mathring{\mathbf{D}}\mathbf{W}], \mathrm{tr}[\mathbf{D}(\mathring{\mathbf{D}})^2\mathbf{W}], \mathrm{tr}[\mathbf{D}\mathbf{W}^2\mathring{\mathbf{D}}\mathbf{W}]. \quad (7)$$

It is clear that at the present time it is certainly impossible and also unrealistic to identify all the coefficients $a_0, a_1, a_2, \ldots, h_4$, and h_5 appearing in Eq. (6), which are functions of the invariants listed in Eq. (7) as well as K and ε, based on the presently available experimental results and the DNS data for turbulence. Thus, we are content with working out an approximate form to Eq. (6) so that it can be practically applied to turbulence modeling.

To this end, first, let us consider the following process of approximation. We shall neglect those terms (I) which are nonlinear in $\mathring{\boldsymbol{\tau}}$ and $\mathring{\mathbf{D}}$; (II) which are cubic or of higher order in form involving $\mathring{\boldsymbol{\tau}}, \mathbf{D}, \mathring{\mathbf{D}}$, and $\mathbf{W}$; and (III) the quadratic terms involving $\mathring{\boldsymbol{\tau}}$, namely, only the linear term of $\mathring{\boldsymbol{\tau}}$ is considered here. Consequently, we arrive at the following approximate form for Eq. (6):

$$\boldsymbol{\tau} = a_0\mathbf{1} + a_1\mathring{\boldsymbol{\tau}} + a_2\mathbf{D} + b_2\mathbf{D}^2 + b_3(\mathring{\mathbf{D}})^2 + b_4\mathbf{W}^2 + e_1[\mathbf{D}\mathring{\mathbf{D}} + \mathring{\mathbf{D}}\mathbf{D}] + g_1[\mathbf{D}\mathbf{W} - \mathbf{W}\mathbf{D}] + h_1[\mathring{\mathbf{D}}\mathbf{W} - \mathbf{W}\mathring{\mathbf{D}}]. \quad (8)$$

Second, here, we shall apply an approximation technique akin to the so-called Maxwellian iteration as done by Huang [2]. In fact, this method was introduced by Truesdell [17] in his studies of kinetic theory of gases and given the name; later, it was used in extended thermodynamics (see Müller and Ruggeri [18]) and introduced to turbulence modeling (see Refs. [2,19]). Thus, we substitute the standard K-ε model (see Launder and Spalding [20])

$$\boldsymbol{\tau} = \frac{2K}{3}\mathbf{1} - 2C_\mu\frac{K^2}{\varepsilon}\mathbf{D}, \quad (9)$$

where $C_\mu = 0.09$, into the right hand side of Eq. (8) for $\mathring{\boldsymbol{\tau}}$ to arrive at (note that $\mathring{\mathbf{1}} = \mathbf{0}$)

$$\boldsymbol{\tau} = a_0\mathbf{1} + a_1\overset{\circ}{\overline{\left[\frac{2K}{3}\mathbf{1} - 2C_\mu\frac{K^2}{\varepsilon}\mathbf{D}\right]}} + a_2\mathbf{D} + a_3\mathring{\mathbf{D}} + b_2\mathbf{D}^2 + b_4\mathbf{W}^2 + e_1[\mathbf{D}\mathring{\mathbf{D}} + \mathring{\mathbf{D}}\mathbf{D}] + g_1[\mathbf{D}\mathbf{W} - \mathbf{W}\mathbf{D}] + h_1[\mathring{\mathbf{D}}\mathbf{W} - \mathbf{W}\mathring{\mathbf{D}}]$$
$$= \left(a_0 + \frac{2}{3}\dot{K}a_1\right)\mathbf{1} + a_2\mathbf{D} + b_2\mathbf{D}^2 + \left(a_3 - 2C_\mu\frac{K^2}{\varepsilon}\right)\mathring{\mathbf{D}} + 2C_\mu a_1\frac{K^2\dot{\varepsilon} - 2K\dot{K}\varepsilon}{\varepsilon^2}\mathbf{D} + b_4\mathbf{W}^2 + e_1[\mathbf{D}\mathring{\mathbf{D}} + \mathring{\mathbf{D}}\mathbf{D}] + g_1[\mathbf{D}\mathbf{W} - \mathbf{W}\mathbf{D}] + h_1[\mathring{\mathbf{D}}\mathbf{W} - \mathbf{W}\mathring{\mathbf{D}}], \quad (10)$$

where for simplicity in notation and without confusion, hereinafter, an overdot is also used to denote the material time derivative D/Dt associated with the mean velocity field $\overline{\mathbf{v}}$.

Note that in Eq. (10) the coefficients a_0, a_1, a_2, a_3, b_4, e_1, and h_1 are functions of the invariants listed in Eq. (7), K and ε, and are independent of one another. Therefore, Eq. (10) can be written as

$$\boldsymbol{\tau} = \alpha_0\mathbf{1} + \alpha_1\mathbf{D} + \alpha_2\mathbf{D}^2 + \alpha_3\mathring{\mathbf{D}} + \alpha_4\frac{K^2\dot{\varepsilon} - 2K\dot{K}\varepsilon}{\varepsilon^2}\mathbf{D} + \alpha_5\mathbf{W}^2 + \alpha_6[\mathbf{D}\mathbf{W} - \mathbf{W}\mathbf{D}] + \alpha_7[\mathring{\mathbf{D}}\mathbf{W} - \mathbf{W}\mathring{\mathbf{D}}] + \alpha_8[\mathbf{D}\mathring{\mathbf{D}} + \mathring{\mathbf{D}}\mathbf{D}], \quad (11)$$

where $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_7$, and α_8 are functions of K, ε and the invariants listed in Eq. (7).

Taking contraction on Eq. (11) and observing that $\mathrm{tr}\mathbf{D} = \mathrm{tr}\mathring{\mathbf{D}} = 0$, we have

$$\mathrm{tr}\boldsymbol{\tau} = 2K = 3\alpha_0 + \alpha_2\mathrm{tr}\mathbf{D}^2 + \alpha_5\mathrm{tr}\mathbf{W}^2 + 2\alpha_8\mathrm{tr}(\mathbf{D}\mathring{\mathbf{D}}). \quad (12)$$

On substitution of Eq. (12) into Eq. (11), it follows that

$$\boldsymbol{\tau} = \frac{2K}{3}\mathbf{1} + \alpha_1\mathbf{D} + \alpha_2\left[\mathbf{D}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{D}^2)\mathbf{1}\right] + \alpha_3\mathring{\mathbf{D}} + \alpha_4\frac{K^2\dot{\varepsilon} - 2K\dot{K}\varepsilon}{\varepsilon^2}\mathbf{D} + \alpha_5\left[\mathbf{W}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{W}^2)\mathbf{1}\right] + \alpha_6[\mathbf{D}\mathbf{W} - \mathbf{W}\mathbf{D}] + \alpha_7[\mathring{\mathbf{D}}\mathbf{W} - \mathbf{W}\mathring{\mathbf{D}}] + \alpha_8\left[\mathbf{D}\mathring{\mathbf{D}} + \mathring{\mathbf{D}}\mathbf{D} - \frac{2}{3}\mathrm{tr}(\mathbf{D}\mathring{\mathbf{D}})\mathbf{1}\right]. \quad (13)$$

In addition, dimensional analysis gives

$$\alpha_1=\beta_1\frac{K^2}{\varepsilon},\quad \alpha_2=\beta_2\frac{K^3}{\varepsilon^2},\quad \alpha_3=\beta_3\frac{K^3}{\varepsilon^2},$$

$$\alpha_4=\beta_4\frac{K}{\varepsilon},\quad \alpha_5=\beta_5\frac{K^3}{\varepsilon^2},\quad \alpha_6=\beta_6\frac{K^3}{\varepsilon^2},$$

$$\alpha_7=\beta_7\frac{K^4}{\varepsilon^3},\quad \alpha_8=\beta_8\frac{K^4}{\varepsilon^3}. \tag{14}$$

With Eq. (14), now Eq. (13) reads

$$\begin{aligned}\boldsymbol{\tau}&=\frac{2K}{3}\mathbf{1}+\beta_1\frac{K^2}{\varepsilon}\mathbf{D}+\beta_2\frac{K^3}{\varepsilon^2}\left[\mathbf{D}^2-\frac{1}{3}\mathrm{tr}(\mathbf{D}^2)\mathbf{1}\right]+\beta_3\frac{K^3}{\varepsilon^2}\mathring{\mathbf{D}}\\&+\beta_4\frac{K^2}{\varepsilon^3}(K\dot{\varepsilon}-2\dot{K}\varepsilon)\mathbf{D}+\beta_5\frac{K^3}{\varepsilon^2}\left[\mathbf{W}^2-\frac{1}{3}\mathrm{tr}(\mathbf{W}^2)\mathbf{1}\right]\\&+\beta_6\frac{K^3}{\varepsilon^2}[\mathbf{DW}-\mathbf{WD}]+\beta_7\frac{K^4}{\varepsilon^3}[\mathring{\mathbf{D}}\mathbf{W}-\mathbf{W}\mathring{\mathbf{D}}]\\&+\beta_8\frac{K^4}{\varepsilon^3}\left[\mathbf{D}\mathring{\mathbf{D}}+\mathring{\mathbf{D}}\mathbf{D}-\frac{2}{3}\mathrm{tr}(\mathbf{D}\mathring{\mathbf{D}})\mathbf{1}\right],\end{aligned}\tag{15}$$

where the dimensionless coefficients $\beta_1,\beta_2,\ldots,\beta_7$, and β_8 are functions of K, ε, and the invariants listed in Eq. (7).

Recall that a nonlinear K-ε model was developed as one example to illustrate the general approach to modeling the Reynolds stress by Huang [2], which is

$$\begin{aligned}\boldsymbol{\tau}&=\frac{2K}{3}\mathbf{1}-2C_\mu\frac{K^2}{\epsilon}\mathbf{D}+\gamma_1C_\mu^2\frac{K^3}{\epsilon^2}\left[\mathbf{D}^2-\frac{1}{3}\mathrm{tr}(\mathbf{D}^2)\mathbf{1}\right]-\gamma_2C_\mu^2\frac{K^3}{\epsilon^2}\mathring{\mathbf{D}}\\&-\gamma_3C_\mu\frac{K^2}{\epsilon^3}(K\dot{\epsilon}-2\dot{K}\epsilon)\mathbf{D},\end{aligned}\tag{16}$$

where $C_\mu=0.09$, $\gamma_1=2.896$, $\gamma_2=2.784$, and $\gamma_3=0.843$. The coefficients γ_1,γ_2, and γ_3 were analytically identified based on the averaged values of the experimental results of Tavoularis and Corrsin [3], Tavoularis and Karnik [21], and the DNS data of Rogers *et al.* [22]. The numerical calculations based on this *quadratic* model given in Ref. [2] for homogeneous turbulent shear flow and fully developed turbulent flow over a backward-facing step (see Yang *et al.* [23]) are comparable with those obtained by employing the model of Shih, Zhu, and Lumley [10] and that of Craft, Launder, and Suga [11], showing a reasonably good agreement with the experimental results of the Reynolds stresses given in Driver and Seegmiller [24], except in the region near the reattachment point in backward-facing step flow.

It is evident that to include Eq. (16) as a special case we can simply set in Eq. (15)

$$\beta_1=-2C_\mu,\quad \beta_2=\gamma_1C_\mu^2,\quad \beta_3=-\gamma_2C_\mu^2,\quad \beta_4=-\gamma_3C_\mu \tag{17}$$

to obtain

$$\begin{aligned}\boldsymbol{\tau}&=\frac{2K}{3}\mathbf{1}-2C_\mu\frac{K^2}{\epsilon}\mathbf{D}+\gamma_1C_\mu^2\frac{K^3}{\epsilon^2}\left[\mathbf{D}^2-\frac{1}{3}\mathrm{tr}(\mathbf{D}^2)\mathbf{1}\right]-\gamma_2C_\mu^2\frac{K^3}{\epsilon^2}\mathring{\mathbf{D}}\\&-\gamma_3C_\mu\frac{K^2}{\epsilon^3}(K\dot{\epsilon}-2\dot{K}\epsilon)\mathbf{D}+\beta_5\frac{K^3}{\varepsilon^2}\left[\mathbf{W}^2-\frac{1}{3}\mathrm{tr}(\mathbf{W}^2)\mathbf{1}\right]\\&+\beta_6\frac{K^3}{\varepsilon^2}[\mathbf{DW}-\mathbf{WD}]+\beta_7\frac{K^4}{\varepsilon^3}[\mathring{\mathbf{D}}\mathbf{W}-\mathbf{W}\mathring{\mathbf{D}}]\\&+\beta_8\frac{K^4}{\varepsilon^3}\left[\mathbf{D}\mathring{\mathbf{D}}+\mathring{\mathbf{D}}\mathbf{D}-\frac{2}{3}\mathrm{tr}(\mathbf{D}\mathring{\mathbf{D}})\mathbf{1}\right].\end{aligned}\tag{18}$$

In the following, we shall determine the coefficients β_5,β_6,β_7, and β_8 according to the experimental results.

C. Identification of the model coefficients

Ever since Prandtl [25] put forth his pioneering mixing-length theory in which for the first time the notion of eddy viscosity introduced by Boussinesq [26] was found of practical application in turbulence modeling, the workers in turbulence modeling have been facing the arduous tasks in identifying the model coefficients for their proposed models—the process of which is usually a painstaking interplay between the theory and the relevant well-devised experiments (see, e.g., Launder and Spalding [27], Gatski and Speziale [9], and Shih *et al.* [10]). With the rapid development of the supercomputers as witnessed in the past two decades, the direct numerical simulation, although still very costly even for relatively low or moderate Reynolds number turbulence simulation at the present stage and also having its own difficulties, now, however, plays a more and more important role in turbulence modeling and provides a valuable means that helps to determine the model coefficients appearing in such as the two-equation K-ε (low Reynolds number) models and in the so-called large-eddy simulation of turbulence (see, e.g., Kim *et al.* [28], Speziale [29], Rodi and Mansour [30], and Kasagi and Shikazono [31]).

Now let us consider the following form of Eq. (15), which includes Eq. (16) as a special case, namely:

$$\begin{aligned}\boldsymbol{\tau}&=\frac{2K}{3}\mathbf{1}+\beta_1\frac{K^2}{\varepsilon}\mathbf{D}+\beta_2\frac{K^3}{\varepsilon^2}\left[\mathbf{D}^2-\frac{1}{3}\mathrm{tr}(\mathbf{D}^2)\mathbf{1}\right]+\beta_3\frac{K^3}{\varepsilon^2}\mathring{\mathbf{D}}\\&+\beta_4\frac{K^2}{\varepsilon^3}(K\dot{\varepsilon}-2\dot{K}\varepsilon)\mathbf{D}+\beta_5\frac{K^3}{\varepsilon^2}\left[\mathbf{W}^2-\frac{1}{3}\mathrm{tr}(\mathbf{W}^2)\mathbf{1}\right]\\&+\beta_6\frac{K^3}{\varepsilon^2}[\mathbf{DW}-\mathbf{WD}]+\beta_7\frac{K^4}{\varepsilon^3}[\mathring{\mathbf{D}}\mathbf{W}-\mathbf{W}\mathring{\mathbf{D}}]\\&+\beta_8\frac{K^4}{\varepsilon^3}\left[\mathbf{D}\mathring{\mathbf{D}}+\mathring{\mathbf{D}}\mathbf{D}-\frac{2}{3}\mathrm{tr}(\mathbf{D}\mathring{\mathbf{D}})\mathbf{1}\right],\end{aligned}\tag{19}$$

where $\beta_1=-2C_\mu$, $\beta_2=\gamma_1C_\mu^2=2.896C_\mu^2$, $\beta_3=-\gamma_2C_\mu^2=-2.784C_\mu^2$, $\beta_4=-\gamma_3C_\mu=-0.843C_\mu$, β_5, β_6, β_7, and β_8 are to be determined.

Here, we shall adopt the experimental results of the fully developed homogeneous turbulent shear flow in Tavoularis and Corrsin [3] to identify the coefficients β_5,β_6,β_7, and β_8. However, since in homogeneous turbulent shear flow $\mathbf{D}\mathring{\mathbf{D}}+\mathring{\mathbf{D}}\mathbf{D}$ reduces to $\mathbf{D}^2\mathbf{W}+\mathbf{W}\mathbf{D}^2=\mathbf{0}$ (note that $\dot{\mathbf{D}}=\mathbf{0}$), the coefficient β_8 is thus not determinable at the moment with the above-mentioned experimental results and will be left for determination in future work. Therefore, we shall consider

$$\boldsymbol{\tau} = \frac{2K}{3}\mathbf{1} + \beta_1\frac{K^2}{\varepsilon}\mathbf{D} + \beta_2\frac{K^3}{\varepsilon^2}\left[\mathbf{D}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{D}^2)\mathbf{1}\right] + \beta_3\frac{K^3}{\varepsilon^2}\mathring{\mathbf{D}} + \beta_4\frac{K^2}{\varepsilon^3}(K\dot{\varepsilon} - 2\dot{K}\varepsilon)\mathbf{D} + \beta_5\frac{K^3}{\varepsilon^2}\left[\mathbf{W}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{W}^2)\mathbf{1}\right] + \beta_6\frac{K^3}{\varepsilon^2}[\mathbf{DW} - \mathbf{WD}] + \beta_7\frac{K^4}{\varepsilon^3}[\mathring{\mathbf{D}}\mathbf{W} - \mathbf{W}\mathring{\mathbf{D}}]. \tag{20}$$

Note that the anisotropy stress tensor is defined as $\mathbf{a}:=[\boldsymbol{\tau}-(2K/3)\mathbf{1}]/2K$. Then Eq. (20) can be written as

$$\mathbf{a} = \frac{\beta_1}{2}\frac{K}{\varepsilon}\mathbf{D} + \frac{\beta_2}{2}\frac{K^2}{\varepsilon^2}\left[\mathbf{D}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{D}^2)\mathbf{1}\right] + \frac{\beta_3}{2}\frac{K^2}{\varepsilon^2}\mathring{\mathbf{D}} + \frac{\beta_4}{2}\frac{K}{\varepsilon^3}(K\dot{\varepsilon} - 2\dot{K}\varepsilon)\mathbf{D} + \frac{\beta_5}{2}\frac{K^2}{\varepsilon^2}\left[\mathbf{W}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{W}^2)\mathbf{1}\right] + \frac{\beta_6}{2}\frac{K^2}{\varepsilon^2}[\mathbf{DW} - \mathbf{WD}] + \frac{\beta_7}{2}\frac{K^3}{\varepsilon^3}[\mathring{\mathbf{D}}\mathbf{W} - \mathbf{W}\mathring{\mathbf{D}}]. \tag{21}$$

Let us consider the fully developed homogeneous turbulent shear flow, in which the gradient of the mean velocity field

$$(\mathrm{grad}\overline{\mathbf{v}}) = \begin{pmatrix} 0 & S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{22}$$

where $S=\mathrm{const}$. It follows that the mean stretching tensor

$$(\mathbf{D}) = \begin{pmatrix} 0 & S/2 & 0 \\ S/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{23}$$

and the mean spin tensor

$$(\mathbf{W}) = \begin{pmatrix} 0 & S/2 & 0 \\ -S/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

Here, it is worth noting that the work of Lee *et al.* [32] reveals interesting similarities in statistical correlations and instantaneous structures between the viscous sublayer of (inhomogeneous) turbulent channel flow and the *homogenous* turbulent shear flow at high shear rate.

The asymptotic values of homogeneous turbulent shear flow in Tavoularis and Corrsin [3] are as follows:

$$a_{11}^{\infty} = 0.197, \quad a_{12}^{\infty} = -0.140, \quad a_{22}^{\infty} = -0.143,$$

$$\left(\frac{SK}{\varepsilon}\right)_{\infty} = 6.25. \tag{25}$$

In homogeneous turbulent shear flow, by Eq. (21), we have for the anisotropy shear stress a_{12}

$$a_{12} = -\frac{\beta_1}{4}\left(\frac{KS}{\varepsilon}\right) + \frac{\beta_4}{2}\frac{K}{\varepsilon^3}(K\dot{\varepsilon} - 2\dot{K}\varepsilon)\frac{S}{2} - \frac{\beta_7}{4}\left(\frac{KS}{\varepsilon}\right)^3 = -\frac{C_\mu}{2}\left(\frac{KS}{\varepsilon}\right) + \frac{\beta_4}{4}\left[a_{12}(4 - 2C_{\varepsilon 1})\left(\frac{KS}{\varepsilon}\right)^2 + (2 - C_{\varepsilon 2})\left(\frac{KS}{\varepsilon}\right)\right] - \frac{\beta_7}{4}\left(\frac{KS}{\varepsilon}\right)^3, \tag{26}$$

in which Eqs. (34) and (35), in this case reduced to

$$\dot{K} = -2KSa_{12} - \varepsilon, \tag{27a}$$

$$\dot{\varepsilon} = -2C_{\varepsilon 1}\varepsilon S a_{12} - C_{\varepsilon 2}\frac{\varepsilon^2}{K}, \tag{27b}$$

have been used.

Taking the asymptotic value of Eq. (26) then yields

$$a_{12}^{\infty} = -\frac{C_\mu}{2}\left(\frac{KS}{\varepsilon}\right)_{\infty} + \frac{\beta_4}{4}\left[a_{12}^{\infty}(4 - 2C_{\varepsilon 1})\left(\frac{KS}{\varepsilon}\right)_{\infty}^2 + (2 - C_{\varepsilon 2}) \times \left(\frac{KS}{\varepsilon}\right)_{\infty}\right] - \frac{\beta_7}{4}\left(\frac{KS}{\varepsilon}\right)_{\infty}^3, \tag{28}$$

where $C_\mu=0.09, C_{\varepsilon 1}=1.44$, and $C_{\varepsilon 2}=1.92$.

In addition, it follows from Eq. (21) that

$$a_{11}^{\infty} = \frac{\beta_2}{24}\left(\frac{KS}{\varepsilon}\right)_{\infty}^2 - \frac{\beta_3}{4}\left(\frac{KS}{\varepsilon}\right)_{\infty}^2 - \frac{\beta_5}{24}\left(\frac{KS}{\varepsilon}\right)_{\infty}^2 - \frac{\beta_6}{4}\left(\frac{KS}{\varepsilon}\right)_{\infty}^2, \tag{29}$$

$$a_{22}^{\infty} = \frac{\beta_2}{24}\left(\frac{KS}{\varepsilon}\right)_{\infty}^2 + \frac{\beta_3}{4}\left(\frac{KS}{\varepsilon}\right)_{\infty}^2 - \frac{\beta_5}{24}\left(\frac{KS}{\varepsilon}\right)_{\infty}^2 + \frac{\beta_6}{4}\left(\frac{KS}{\varepsilon}\right)_{\infty}^2. \tag{30}$$

By Eq. (25) together with Eqs. (28)–(30), and noting that $\beta_2=\gamma_1 C_\mu^2=2.896C_\mu^2$, $\beta_3=-\gamma_2 C_\mu^2=-2.784C_\mu^2$, $\beta_4=-\gamma_3 C_\mu=-0.843C_\mu$, we obtain

$$\beta_5 = \gamma_4 C_\mu^2 = 0.8482C_\mu^2, \quad \beta_6 = \gamma_5 C_\mu^2 = 0.6344C_\mu^2,$$

$$\beta_7 = -\gamma_6 C_\mu^2 = -0.7767C_\mu^3. \tag{31}$$

Finally, we arrive at the following nonlinear K-ε model which adopts the Jaumann derivative:

$$\boldsymbol{\tau} = \frac{2K}{3}\mathbf{1} + \beta_1\frac{K^2}{\varepsilon}\mathbf{D} + \beta_2\frac{K^3}{\varepsilon^2}\left[\mathbf{D}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{D}^2)\mathbf{1}\right] + \beta_3\frac{K^3}{\varepsilon^2}\mathring{\mathbf{D}} + \beta_4\frac{K^2}{\varepsilon^3}(K\dot{\varepsilon} - 2\dot{K}\varepsilon)\mathbf{D} + \beta_5\frac{K^3}{\varepsilon^2}\left[\mathbf{W}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{W}^2)\mathbf{1}\right] + \beta_6\frac{K^3}{\varepsilon^2}[\mathbf{DW} - \mathbf{WD}] + \beta_7\frac{K^4}{\varepsilon^3}[\mathring{\mathbf{D}}\mathbf{W} - \mathbf{W}\mathring{\mathbf{D}}], \tag{32}$$

where $\beta_1=-2C_\mu$, $\beta_2=\gamma_1 C_\mu^2=2.896C_\mu^2$, $\beta_3=-\gamma_2 C_\mu^2=-2.784C_\mu^2$, $\beta_4=-\gamma_3 C_\mu=-0.843C_\mu$, $\beta_5=\gamma_4 C_\mu^2=0.8482C_\mu^2$, $\beta_6=\gamma_5 C_\mu^2=0.6344C_\mu^2$, and $\beta_7=-\gamma_6 C_\mu^3=-0.7767C_\mu^3$. Note that, due to the involvement of $\mathring{\mathbf{D}}$, the Jaumann derivative of $\mathbf{D}$, the last term is in fact a cubic term.

Moreover, if the Jaumann derivative is replaced by the Oldroyd derivative, then, after going through the same procedure as above based on the experimental results of Tavoularis and Corrsin [3], we arrive at the following model which extends the one given by Huang [2] to include the contribution of the mean spin tensor **W**:

$$\boldsymbol{\tau} = \frac{2K}{3}\mathbf{1} + \beta_1\frac{K^2}{\varepsilon}\mathbf{D} + \beta_2\frac{K^3}{\varepsilon^2}\left[\mathbf{D}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{D}^2)\mathbf{1}\right] + \beta_3\frac{K^3}{\varepsilon^2}\left[\overset{\diamond}{\mathbf{D}} - \frac{1}{3}\mathrm{tr}(\overset{\diamond}{\mathbf{D}})\mathbf{1}\right] + \beta_4\frac{K^2}{\varepsilon^3}(K\dot{\varepsilon} - 2\dot{K}\varepsilon)\mathbf{D} + \beta_5\frac{K^3}{\varepsilon^2}\left[\mathbf{W}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{W}^2)\mathbf{1}\right] + \beta_6\frac{K^3}{\varepsilon^2}[\mathbf{DW} - \mathbf{WD}] + \beta_7\frac{K^4}{\varepsilon^3}[\overset{\diamond}{\mathbf{D}}\mathbf{W} - \mathbf{W}\overset{\diamond}{\mathbf{D}}], \tag{33}$$

where $\beta_1 = -2C_\mu$, $\beta_2 = -4\sigma_1 C_\mu^2 = -8.012C_\mu^2$, $\beta_3 = -4\sigma_2 C_\mu^2 = -5.568C_\mu^2$, $\beta_4 = -\sigma_3 C_\mu = -0.843C_\mu$, $\beta_5 = \sigma_4 C_\mu^2 = 1.075C_\mu^2$, $\beta_6 = \sigma_5 C_\mu^2 = 3.419C_\mu^2$, and $\beta_7 = -\sigma_6 C_\mu^3 = -0.7767C_\mu^3$. Here, $\diamond$ denotes the Oldroyd derivative; $\overset{\diamond}{\mathbf{D}} = D\mathbf{D}/Dt - (\mathrm{grad}\overline{\mathbf{v}})\mathbf{D} - \mathbf{D}(\mathrm{grad}\overline{\mathbf{v}})^T$. As in Eq. (32), because of the involvment of $\overset{\diamond}{\mathbf{D}}$, the last term is in fact a cubic term as well.

III. NUMERICAL RESULTS OF TWO BENCHMARK TURBULENT FLOWS

In the following numerical calculations, the conventional modeled K equation and ε equation will be used, which are

$$\dot{K} = -\tau_{ij}\frac{\partial \overline{v_i}}{\partial x_j} - \varepsilon + \frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_K}\frac{\partial K}{\partial x_i}\right) + \nu\nabla^2 K, \tag{34}$$

$$\dot{\varepsilon} = -C_{\varepsilon 1}\frac{\varepsilon}{K}\tau_{ij}\frac{\partial \overline{v_i}}{\partial x_j} - C_{\varepsilon 2}\frac{\varepsilon^2}{K} + \frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) + \nu\nabla^2\varepsilon, \tag{35}$$

where $\nu_T = C_\mu K^2/\varepsilon$, $C_\mu = 0.09$, $C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.92$, $\sigma_K = 1.0$, and $\sigma_\varepsilon = 1.3$. However, it should be noted that modeling the ε equation, due to having more unknown terms that need to be modeled, thus, poses more severe difficulties than modeling the K equation, which is not only the trace of the Reynolds stress transport equation but also, importantly, a direct consequence of the first law of thermodynamics as shown in Huang and Durst [33]. In fact, both Eq. (34) and Eq. (35), especially the latter, need to be modified so as to better predict the complex turbulent flows (see, e.g., Launder and Spalding [27], Speziale [29], and Yoshizawa [34]). Although this is not our main concern in this work, we should keep in mind that, if any modifications were made to the conventional modeled K equation (34) and ε equation (35), all the coefficients appearing in the models (15) and (16) should be reidentified correspondingly according to the experimental results.

We shall carry out numerical calculations based on seven linear and nonlinear K-ε models and make a comparison with the experimental results of fully developed turbulent flow over a backward-facing step in Driver and Seegmiller [24] as well as the DNS data of homogeneous turbulent shear flow in Matsumoto *et al.* [35], respectively.

(I) The standard K-ε model (SKE) [20]:

$$\boldsymbol{\tau} = \frac{2K}{3}\mathbf{1} - 2C_\mu\frac{K^2}{\varepsilon}\mathbf{D}, \tag{36}$$

where $C_\mu = 0.09$.

(II) The nonlinear quadratic K-ε model of Speziale [36]:

$$\boldsymbol{\tau} = \frac{2K}{3}\mathbf{1} - 2C_\mu\frac{K^2}{\varepsilon}\mathbf{D} + 4C_D C_\mu^2\frac{K^3}{\varepsilon^2}\left[\mathbf{D}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{D}^2)\mathbf{1}\right] + 4C_E C_\mu^2\frac{K^3}{\varepsilon^2}\left[\overset{\diamond}{\mathbf{D}} - \frac{1}{3}\mathrm{tr}(\overset{\diamond}{\mathbf{D}})\mathbf{1}\right], \tag{37}$$

where $C_\mu = 0.09$, $C_D = C_E = 1.68$, and $\overset{\diamond}{\mathbf{D}} = \dot{\mathbf{D}} - (\mathrm{grad}\overline{\mathbf{v}})\mathbf{D} - \mathbf{D}(\mathrm{grad}\overline{\mathbf{v}})^T$ is the Oldroyd derivative of **D**.

(III) The nonlinear cubic model of Craft, Launder, and Suga (CLS) [11]:

$$\boldsymbol{\tau} = \frac{2K}{3}\mathbf{1} - 2\tilde{C}_\mu\frac{K^2}{\tilde{\varepsilon}}\mathbf{D} + \beta_1\frac{K^3}{\tilde{\varepsilon}^2}\left[\mathbf{D}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{D}^2)\mathbf{1}\right] + \beta_2\frac{K^3}{\tilde{\varepsilon}^2}(\mathbf{WD} - \mathbf{DW}) + \beta_3\frac{K^3}{\tilde{\varepsilon}^2}\left[\mathbf{W}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{W}^2)\mathbf{1}\right] - \gamma_1\frac{K^4}{\tilde{\varepsilon}^3}\mathrm{tr}(\mathbf{D}^2)\mathbf{D} - \gamma_2\frac{K^4}{\tilde{\varepsilon}^3}\mathrm{tr}(\mathbf{W}^2)\mathbf{D} - \gamma_3\frac{K^4}{\tilde{\varepsilon}^3}\left[\mathbf{W}^2\mathbf{D} + \mathbf{DW}^2 - \frac{2}{3}\mathrm{tr}(\mathbf{W}^2\mathbf{D})\mathbf{1}\right] - \gamma_4\frac{K^4}{\tilde{\varepsilon}^3}(\mathbf{WD}^2 - \mathbf{D}^2\mathbf{W}), \tag{38}$$

where $\tilde{\varepsilon}$ is the isotropic dissipation rate.

$$\tilde{C}_\mu = \frac{0.3\{1 - \exp[-0.36/\exp(-0.75\eta)]\}}{1 + 0.35\eta^{3/2}}, \quad \eta = \max(\tilde{S}, \tilde{\Omega}),$$

in which

$$\tilde{S} = (K/\tilde{\varepsilon})[2\mathrm{tr}(\mathbf{D}^2)]^{1/2}, \quad \tilde{\Omega} = (K/\tilde{\varepsilon})[-2\mathrm{tr}(\mathbf{W}^2)]^{1/2},$$

and $\beta_1 = -0.4\tilde{C}_\mu$, $\beta_2 = 0.4\tilde{C}_\mu$, $\beta_3 = -1.04\tilde{C}_\mu$, $\gamma_1 = \gamma_2 = 40.0\tilde{C}_\mu^3$, $\gamma_3 = 0$, and $\gamma_4 = -80.0\tilde{C}_\mu^3$. Note that, since $\gamma_3 = 0$, the cubic term with $(\mathbf{W}^2\mathbf{D} + \mathbf{DW}^2)$ in fact makes no contribution in numerical simulations.

(IV) A nonlinear quadratic K-ε model (using the Jaumann derivative) given by Huang [2]:

$$\boldsymbol{\tau} = \frac{2K}{3}\mathbf{1} - 2C_\mu\frac{K^2}{\epsilon}\mathbf{D} + \gamma_1 C_\mu^2\frac{K^3}{\epsilon^2}\left[\mathbf{D}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{D}^2)\mathbf{1}\right] - \gamma_2 C_\mu^2\frac{K^3}{\epsilon^2}\overset{\circ}{\mathbf{D}} - \gamma_3 C_\mu\frac{K^2}{\epsilon^3}(K\dot{\epsilon} - 2\dot{K}\epsilon)\mathbf{D}, \tag{39}$$

where $C_\mu = 0.09$, $\gamma_1 = 2.896$, $\gamma_2 = 2.784$, and $\gamma_3 = 0.843$.

(V) A nonlinear quadratic K-ε model (using the Oldroyd derivative) given by Huang [2]:

$$\boldsymbol{\tau} = \frac{2K}{3}\mathbf{1} - 2C_\mu \frac{K^2}{\varepsilon}\mathbf{D} - 4\sigma_1 C_\mu^2 \frac{K^3}{\varepsilon^2}\left[\mathbf{D}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{D}^2)\mathbf{1}\right] - 4\sigma_2 C_\mu^2 \frac{K^3}{\varepsilon^2}\left[\overset{\diamond}{\mathbf{D}} - \frac{1}{3}(\mathrm{tr}\overset{\diamond}{\mathbf{D}})\mathbf{1}\right] - \sigma_3 C_\mu \frac{K^2}{\varepsilon^3}(K\dot{\varepsilon} - 2\dot{K}\varepsilon)\mathbf{D}, \tag{40}$$

where $C_\mu = 0.09, \sigma_1 = 2.003, \sigma_2 = 1.392, \sigma_3 = 0.843$.

(VI) The present model employing the Jaumann derivative:

$$\boldsymbol{\tau} = \frac{2K}{3}\mathbf{1} - 2C_\mu \frac{K^2}{\varepsilon}\mathbf{D} + \gamma_1 C_\mu^2 \frac{K^3}{\varepsilon^2}\left[\mathbf{D}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{D}^2)\mathbf{1}\right] - \gamma_2 C_\mu^2 \frac{K^3}{\varepsilon^2}\mathring{\mathbf{D}} - \gamma_3 C_\mu \frac{K^2}{\varepsilon^3}(K\dot{\varepsilon} - 2\dot{K}\varepsilon)\mathbf{D} + \gamma_4 C_\mu^2 \frac{K^3}{\varepsilon^2}\left[\mathbf{W}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{W}^2)\mathbf{1}\right] + \gamma_5 C_\mu^2 \frac{K^3}{\varepsilon^2}[\mathbf{DW} - \mathbf{WD}] - \gamma_6 C_\mu^3 \frac{K^4}{\varepsilon^3}[\mathring{\mathbf{D}}\mathbf{W} - \mathbf{W}\mathring{\mathbf{D}}], \tag{41}$$

where $C_\mu = 0.09, \gamma_1 = 2.896, \gamma_2 = 2.784, \gamma_3 = 0.843, \gamma_4 = 0.8482, \gamma_5 = 0.6344$, and $\gamma_6 = 0.7767$. Clearly, this model has generalized, in the sense of refining, the model (39) to be in cubic order by including the contributions of *the mean spin tensor* $\mathbf{W}$.

(VII) The present model employing the Oldroyd derivative:

$$\boldsymbol{\tau} = \frac{2K}{3}\mathbf{1} - 2C_\mu \frac{K^2}{\varepsilon}\mathbf{D} - 4\sigma_1 C_\mu^2 \frac{K^3}{\varepsilon^2}\left[\mathbf{D}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{D}^2)\mathbf{1}\right] - 4\sigma_2 C_\mu^2 \frac{K^3}{\varepsilon^2}\left[\overset{\diamond}{\mathbf{D}} - \frac{1}{3}(\mathrm{tr}\overset{\diamond}{\mathbf{D}})\mathbf{1}\right] - \sigma_3 C_\mu \frac{K^2}{\varepsilon^3}(K\dot{\varepsilon} - 2\dot{K}\varepsilon)\mathbf{D} + \sigma_4 C_\mu^2 \frac{K^3}{\varepsilon^2}\left[\mathbf{W}^2 - \frac{1}{3}\mathrm{tr}(\mathbf{W}^2)\mathbf{1}\right] + \sigma_5 C_\mu^2 \frac{K^3}{\varepsilon^2}[\mathbf{DW} - \mathbf{WD}] - \sigma_6 C_\mu^3 \frac{K^4}{\varepsilon^3}[\overset{\diamond}{\mathbf{D}}\mathbf{W} - \mathbf{W}\overset{\diamond}{\mathbf{D}}], \tag{42}$$

where $C_\mu = 0.09, \sigma_1 = 2.003, \sigma_2 = 1.392, \sigma_3 = 0.843, \sigma_4 = 1.075, \sigma_5 = 3.419$, and $\sigma_6 = 0.7767$. Here, $\diamond$ denotes the Oldroyd derivative, $\overset{\diamond}{\mathbf{D}} = D\mathbf{D}/Dt - (\mathrm{grad}\overline{\mathbf{v}})\mathbf{D} - \mathbf{D}(\mathrm{grad}\overline{\mathbf{v}})^T$. Clearly, Eq. (42) includes Eq. (40) as a special case.

A. Homogeneous turbulent shear flow

It has been well recognized that the homogeneous turbulent shear flow is a simple but critical test case for any newly proposed model to better capture the complex turbulent flows encountered in engineering practice. Here, the fourth-order Runge-Kutta method has been used to calculate the evolution of the turbulent kinetic energy K with the dimensionless time $t^* = St$ (also called the *total shear*), where S is the mean shear rate. The numerical results based on the above models are presented in Fig. 1 in comparison with the DNS data of Matsumoto *et al.* [35] (note that the mean shear given by Matsumoto *et al.* [35] is $S = \partial U / \partial y = 30\ \mathrm{s}^{-1}$, whereas in the experiments of Tavoularis and Corrsin [3] $S = 46.8\ \mathrm{s}^{-1}$). The initial condition of $K_0 = 10.148\,05$ and $\varepsilon_0 = 62.8489$ at $t^* = 1$ is taken from the DNS data of Matsumoto *et al.* [35].

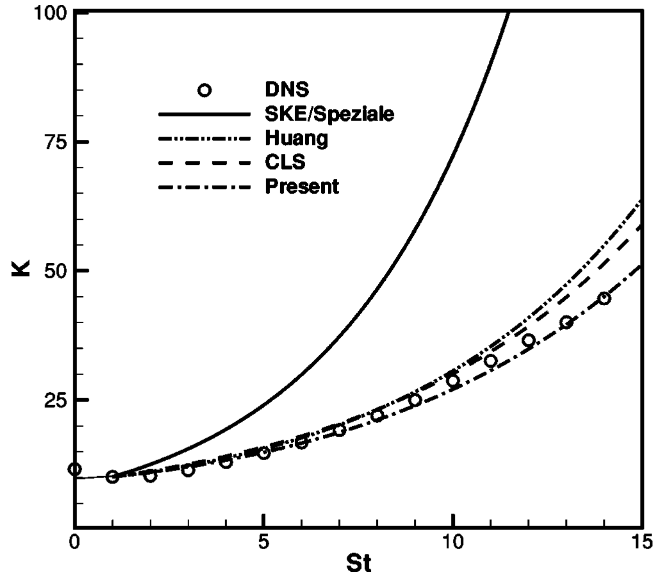

FIG. 1. Time evolution of the turbulent kinetic energy K.

It is seen from Fig. 1 that the result obtained based on the present model shows a good improvement over that predicted by the model of Huang [2] and, in addition, in a good agreement with the trend of the DNS data, a bit better than the result predicted by the model of Craft, Launder, and Suga [11]. Since in this case the anisotropy shear stress a_{12} is the only stress that contributes to modeling the time evolution of the turbulent kinetic energy K, as can be seen from Eqs. (27a) and (27b), the present models (41) and (42), giving rise to the same a_{12}, produce the same result for the time evolution of K. It is interesting to observe that, in this case, in the cubic model of Craft, Launder, and Suga [11], the linear term with $\tilde{C}_\mu$ actually is the only term that contributes to modeling a_{12}, whereas in the present cubic model (41), the cubic term with coefficient γ_6 combines the linear terms with coefficients C_μ and γ_3 to make a joint contribution to modeling the anisotropy shear stress a_{12}; and, similarly, the cubic model (42), which employs a different objective derivative, the Oldroyd derivative, does the same in this regard.

B. Fully developed turbulent flow over a backward-facing step

The fully developed turbulent flow over a backward-facing step, especially with a high ratio of the step height to the tunnel exit height, is another benchmark test case for the accuracy of the closure models in predicting the reattachment location and the skin friction coefficient distribution along the tunnel. In this case (see Fig. 2), we shall carry out the corresponding numerical calculations based on the above seven linear and nonlinear K-ε models (I–VII) and compare the results with the experiments of Driver and Seegmiller [24], in which the geometry has a step height (H) to tunnel exit height ratio of 1:9 and the Reynolds number based on the step height and the experimental reference free-stream velocity is 37423.

Here, the same code based on the finite volume method with nonorthogonal grids (see, e.g., [37]) has been used in

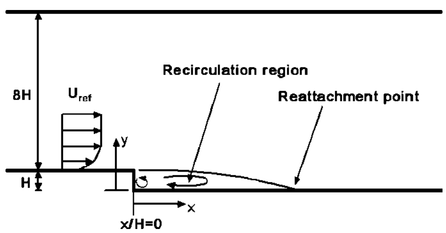

FIG. 2. Schematic of the turbulent flow over a backward-facing step.

our computation. Variable storage is colocated and cell-centered, with Rhie-Chow interpolation for cell-face mass fluxes. The SIMPLE pressure-correction algorithm is used to obtain the pressure field. The convection and diffusion terms of all the equations, including the mean momentum equations and the modeled transport equations for turbulence quantities, are approximated by the second-order central differencing scheme. In addition, the deferred correction technique is used for the discretization of the convection term. Stone's (strongly implicit procedure) method is employed with under-relaxation factors and, in order to stabilize the iteration, a time marching process has been adopted. Convergence is judged by monitoring the magnitude of the absolute residual sources of mass and momentum, normalized by the respective inlet fluxes. The iteration is continued until all the residuals fall below 0.05%. The grid independence was tested first and then a grid of 168×122 adopted in our computation. In the following, a table of the predicted reattachment points is given and then follow the figures of the skin friction coefficient (C_f) distribution and the computed streamlines based on the models I–VII, respectively.

From Table I and Figs. 3, 4, 5, and 6, it is seen that the present cubic models (41) and (42) have produced better results than do the previously developed quadratic models (39) and (40). In particular, it appears that the present model (42) adopting the Oldroyd derivative predicts the reattachment point in closer agreement with the experiment than the prediction by the model of Craft, Launder, and Suga [11]. However, it should be noted that, since in our computation the standard wall function has been used, which usually applies to turbulent attached flow such as the fully developed turbulent channel flow, it would be better to develop the corresponding low Reynolds number versions of the present cubic models (41) and (42) (see, e.g., Rodi and Mansour [30]), so that they can be feasibly applied to modeling complex turbulent flows. In addition, a *generalized* wall function for three-dimensional turbulence recently proposed by Shih *et al.* [38] may be used as well to deal with turbulent separated flows in which cases separation and reattachment occur in company with an adverse pressure gradient appearing in the boundary layer.

TABLE I. Comparison of the predicted reattachment points with the experiment [24].

Model	Reattachment point (x/H)
SKE	5.24
Speziale	5.55
Huang (Jaumann derivative)	5.88
Huang (Oldroyd derivative)	6.04
CLS	6.11
Present (Jaumann derivative)	6.02
Present (Oldroyd derivative)	6.21
Experiment	6.26

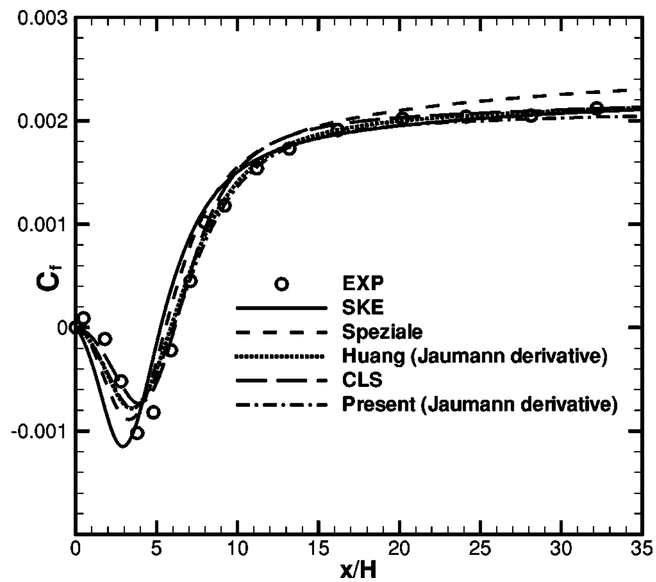

FIG. 3. Skin friction coefficient C_f distribution (I).

IV. CONCLUDING REMARKS

In this work, some history effects have been taken into account to develop a nonlinear K-ε model in the forms of Eqs. (41) and (42), by making use of two objective derivatives, the Jaumann derivative and the Oldroyd derivative, respectively. Like the models of Pope [8], Gatski and Spe-

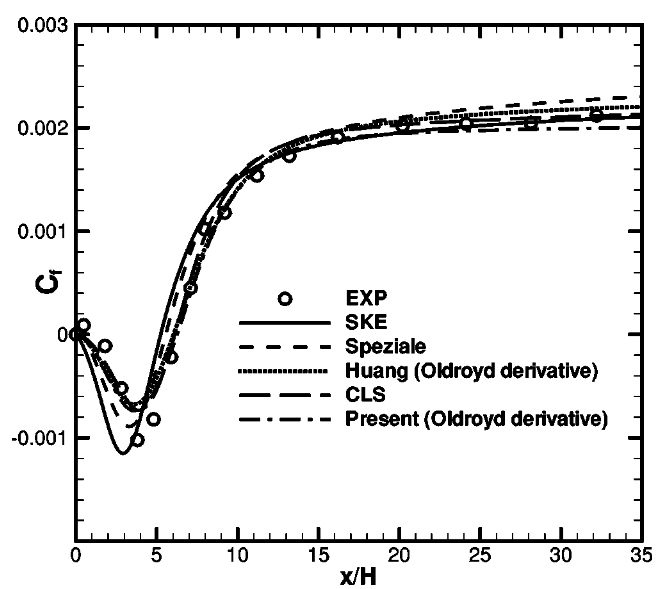

FIG. 4. Skin friction coefficient C_f distribution (II).

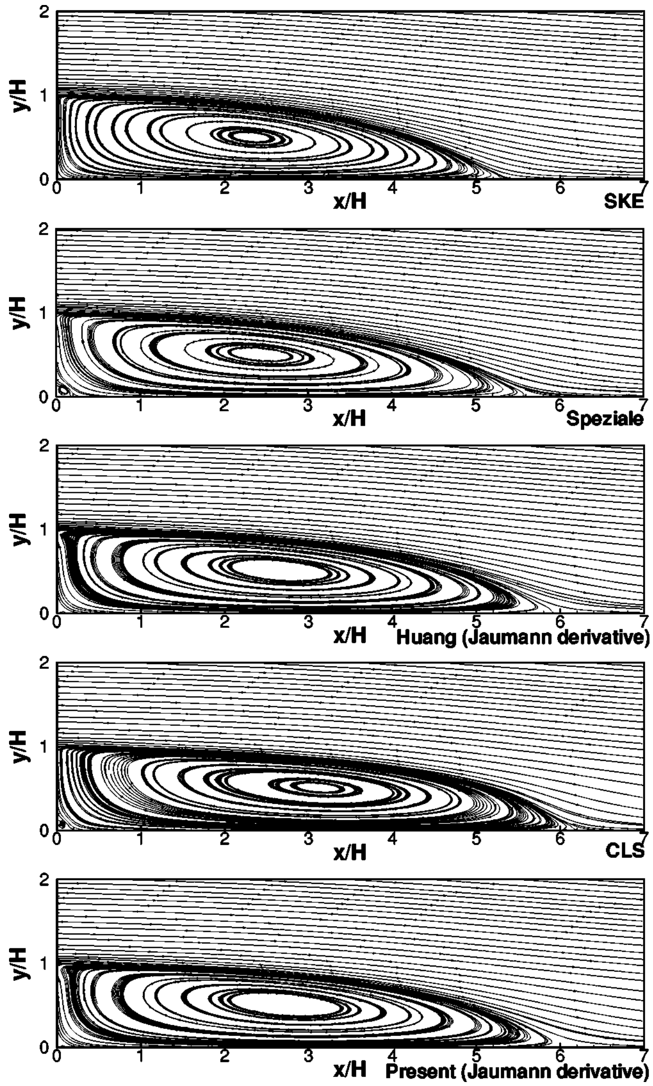

FIG. 5. Computed streamlines (I).

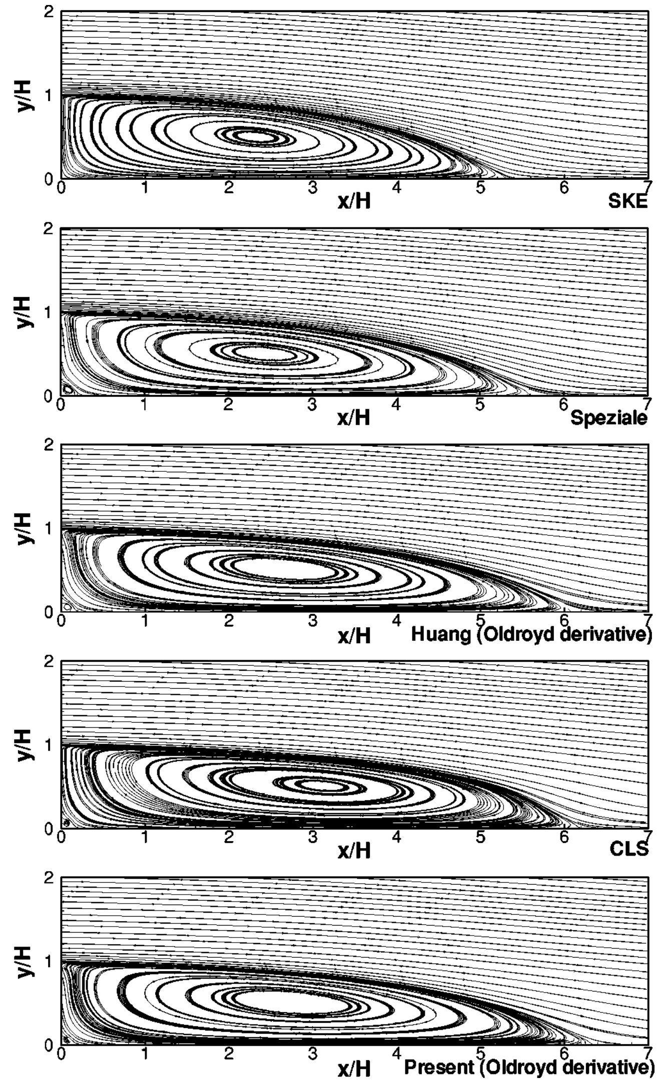

FIG. 6. Computed streamlines (II).

ziale [9], Shih, Zhu, and Lumley [10], Craft, Launder, and Suga [11], and others, this cubic model in the forms of Eqs. (41) and (42) is obviously frame-dependent, as can be seen from the involvement of the mean spin tensor **W**. However, both Eqs. (41) and (42) are but an approximation to the constitutive equation (6), at the complexity level of $p=1, m=1$, and $n=0$ of Eq. (4). In addition, since only one cubic term, i.e., $-\gamma_6 C_\mu^3(K^4/\varepsilon^3)[\overset{\circ}{\mathbf{D}}\mathbf{W}-\mathbf{W}\overset{\circ}{\mathbf{D}}]$ and $-\sigma_6 C_\mu^3(K^4/\varepsilon^3)[\overset{\diamond}{\mathbf{D}}\mathbf{W}-\mathbf{W}\overset{\diamond}{\mathbf{D}}]$, has been included into Eq. (41) and Eq. (42), respectively, it would be interesting to assess the contributions of those formally quadratic, cubic, and higher order terms that have been truncated from Eq. (6) while making the approximations. Nevertheless, the improvement made by this cubic model over the nonlinear quadratic K-ε models given by Huang [2] in predicting the time evolution of the turbulent kinetic energy K in homogeneous turbulent shear flow as well as in turbulent backward-facing step flow is further confirmation, as has been shown by a number of previously proposed quadratic and cubic K-ε models in the literature, of the practical contribution and effectiveness of the mean spin tensor **W** in turbulence modeling. Furthermore, numerical simulations of a number of typical turbulent flows of practical and scientific interest, e.g., the fully developed turbulent flow in an axially rotating pipe (see, e.g., Shih *et al.* [39] and Yang and Ma [40]) and the turbulent secondary flow in a straight tube of noncircular cross section (see, e.g., Bradshaw [41] and Huang and Rajagopal [42]), should be carried out in future work so as to test and then, perhaps, modify the present models for possible broader applications, yet aiming to further develop better closure models for turbulence.

During the development of this new nonlinear K-ε model in the forms of Eqs. (41) and (42), respectively, the Maxwellian iteration method (see Refs. [2,17,19]) has been used to make the relevant approximations to a rate-type closure model for the Reynolds stress (6), as was done in Eq. (8); this method appears to be practically useful in developing the algebraic nonlinear K-ε closure models for turbulence. Moreover, it should be noted that, in order to better capture the complex turbulent phenomena, it is appropriate to appeal to more sophisticated models yet to be explored, say, a nonlinear K-ε model containing more effective quadratic and cubic terms or a rate-type closure model capable of capturing the relaxation effect of the Reynolds stress, which is akin to the second order (moment) closure approach based on modeling the Reynolds stress transport equation, although its implementation in numerical simulations of turbulence at the present time requires a much greater amount of computer resources than the computationally more feasible algebraic nonlinear K-ε models.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China (Grant No. 90205002). We thank H.-J. Chu and X.-D. Yang for assistance with the numerical computations.

[1] R. S. Rivlin, Q. Appl. Math. **15**, 212 (1957).
[2] Y.-N. Huang, Commun. Nonlinear Sci. Numer. Simul. **9**, 543 (2004).
[3] S. Tavoularis and S. Corrsin, J. Fluid Mech. **104**, 311 (1981).
[4] H. W. Liepmann, in *Méchanique de la Turbulence* (CNRS, Paris, 1961).
[5] I. Proudman, J. Fluid Mech. **44**, 563 (1970).
[6] J. L. Lumley, J. Fluid Mech. **41**, 413 (1970).
[7] Y.-N. Huang and F. Durst, Phys. Rev. E **63**, 056305 (2001).
[8] S. B. Pope, J. Fluid Mech. **72**, 331 (1975).
[9] T. B. Gatski and C. G. Speziale, J. Fluid Mech. **254**, 59 (1993).
[10] T.-H. Shih, J. Zhu, and J. L. Lumley, Comput. Methods Appl. Mech. Eng. **125**, 287 (1995).
[11] T. J. Craft, B. E. Launder, and K. Suga, Int. J. Heat Fluid Flow **17**, 108 (1996).
[12] U. Schumann, Phys. Fluids **20**, 721 (1977).
[13] J. L. Lumley, Adv. Appl. Mech. **18**, 123 (1978).
[14] C.-C. Wang, Arch. Ration. Mech. Anal. **36**, 166 (1970).
[15] G. F. Smith, Int. J. Eng. Sci. **9**, 889 (1970).
[16] A. J. M. Spencer, in *Continuum Physics I*, edited by A. C. Eringen (Academic Press, New York, 1971), pp. 292–307.
[17] C. Truesdell, J. Ration. Mech. Anal. **5**, 55 (1956).
[18] I. Müller and T. Ruggeri, *Rational Extended Thermodynamics* (Springer-Verlag, Berlin, 1998).
[19] Y.-N. Huang, Ph.D. dissertation, University of Pittsburgh, 1994.
[20] B. E. Launder and D. B. Spalding, Comput. Methods Appl. Mech. Eng. **3**, 269 (1974).
[21] S. Tavoularis and U. Karnik, J. Fluid Mech. **204**, 457 (1989).
[22] M. M. Rogers, P. Moin, and W. C. Reynolds, Department of Mechanical Engineering Technical Report No. TF-25, Stanford University, 1986 (unpublished).
[23] X.-D. Yang, H.-Y. Ma, and Y.-N. Huang, Commun. Nonlinear Sci. Numer. Simul. (to be published).
[24] D. M. Driver and H. L. Seegmiller, AIAA J. **23**, 163 (1985).
[25] L. Prandtl, Z. Angew. Math. Mech. **5**, 136 (1925).
[26] J. Boussinesq, Mem. Prés. Div. Savants Acad. Sci. Paris **23**, 46 (1877).
[27] B. E. Launder and D. B. Spalding, *Mathematical Models of Turbulence* (Academic Press, London, 1972).
[28] J. Kim, P. Moin, and R. Moser, J. Fluid Mech. **177**, 133 (1987).
[29] C. G. Speziale, Annu. Rev. Fluid Mech. **23**, 107 (1991).
[30] W. Rodi and N. N. Mansour, J. Fluid Mech. **250**, 509 (1993).
[31] N. Kasagi and N. Shikazono, Proc. R. Soc. London, Ser. A **451**, 257 (1995).
[32] M. J. Lee, J. Kim, and P. Moin, J. Fluid Mech. **216**, 561 (1990).
[33] Y.-N. Huang and F. Durst, Int. J. Heat Fluid Flow **22**, 495 (2001).
[34] A. Yoshizawa, Phys. Rev. E **49**, 4065 (1994).
[35] A. Matsumoto, Y. Nagano, and M. Tagawa, in *Proceedings of the 5th Symposium on Computational Fluid Dynamics of the Japan Society of Computational Fluid Dynamics, Tokyo*, 1991, pp. 361–364.
[36] C. G. Speziale, J. Fluid Mech. **178**, 459 (1987).
[37] J. H. Ferziger and M. Perić, *Computational Methods for Fluid Dynamics*, Chap. 7 (Springer-Verlag, Berlin, 1996).
[38] T.-H. Shih, L. A. Povinelli, and N.-S. Liu, J. Turbul. **4**, 15 (2003).
[39] T.-H. Shih, J. Zhu, W. W. Liou, K. H. Chen, N. S. Liu, and J. L. Lumley, in *Proceedings of the 11th Symposium on Turbulent Shear Flows, Grenoble, France*, 1997, pp. 31.1–31.6.
[40] X.-D. Yang and H.-Y. Ma, Int. J. Numer. Methods Fluids **43**, 1355 (2003).
[41] P. Bradshaw, Annu. Rev. Fluid Mech. **19**, 53 (1987).
[42] Y.-N. Huang and K. R. Rajagopal, Math. Models Meth. Appl. Sci. **5**, 111 (1995).